1. 1D Heat equation

Suppose that a smooth solution $u(x,t)$ satisfies the following differential equation

$$u_t = u_{xx} \qquad (\textit{Heat equation}), \tag{1}$$

in $\{(x,t) : 0 \leqslant x \leqslant L, 0 \leqslant t\}$. Then, $u(x,t)$ can represent the temperature under the heat flow on a rod located in $\{0 \leqslant x \leqslant L\}$. In order to solve the equation, we need the initial data

$$u(x,0) = g(x) \qquad (\textit{Cauchy condition}), \tag{2}$$

and one of the following boundary data

$$u(0,t) = h_1(t), \quad u(L,t) = h_2(t) \qquad (\textit{Dirichlet condition}), \tag{3}$$

$$-u_x(0,t) = h_1(t), \quad u_x(L,t) = h_2(t) \qquad (\textit{Neumann condition}), \tag{4}$$

$$-u_x(0,t) + \alpha u(0,t) = h_1(t), \quad u_x(L,t) + \alpha u(L,t) = h_2(t) \qquad (\textit{Robin condition}). \tag{5}$$

2. Uniqueness

We establish the following uniqueness theorem.

Theorem 1 (Uniqueness)**.** *Given smooth functions $g(x), h_1(t), h_2(t)$, the heat equation* (1) *has at most one smooth solution $u(x,t)$ satisfying* (2) *on $\{0 \leqslant x \leqslant L\}$ and* (3) *on $\{t \geqslant 0\}$.*

Proof. Suppose that $u(x,t)$ and $v(x,t)$ are solutions satisfying the conditions. Then, the smooth function $w(x,t) = u(x,t) - v(x,t)$ satisfies

$$w_t = u_t - v_t = u_{xx} - v_{xx} = w_{xx}. \tag{6}$$

Moreover, we can observe

$$w(0,t) = w(L,t) = 0. \tag{7}$$

Next, we define an energy

$$E(t) = \int_0^L w^2(x,t)dx.$$

Then, (6) shows

$$\tfrac{d}{dt}E(t) = \int_0^L 2ww_t dx = 2\int_0^L ww_{xx} dx.$$

Using the integration by part and (7),

$$\tfrac{d}{dt}E(t) = 2ww_x|_0^L - 2\int_0^L |w_x|^2 dx = -2\int_0^L |w_x|^2 dx \leqslant 0. \tag{8}$$

Therefore,

$$0 \leqslant E(t) \leqslant E(0), \tag{9}$$

for all $t \geqslant 0$. However, we have $w(x,0) = 0$ by definition, namely $E(0) = 0$. Thus, $E(t) = 0$ and $w(x,t) = 0$. Hence, the smooth solution is unique. □

Remark. If $h_1(t) = h_2(t) = 0$, then we can modify the proof above to show

$$\frac{d}{dt}\int_0^L u^2(x,t)dx \leqslant 0. \tag{10}$$

Then, it would be a natural question to prove $\lim_{t\to+\infty} \sup_{0\leqslant x\leqslant L} |u(x,t)| = 0$. We will prove this next week, but it'd be good to try to prove it yourself.

3. Review: Fourier series

We recall the Fourier series. In this class, we will use the following fact without proofs.

Given a smooth function $f : [-L, L] \to \mathbb{R}$ with $f(-L) = f(L)$, the following holds

$$\lim_{N\to+\infty} \sup_{|x|\leqslant L} |f(x) - S_N(x)| = 0,$$

for the partial sums $S_N(x)$ of Fourier series,

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L) + \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \qquad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx, \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function satisfying $f(0) = 0$. Then,

$$\lim_{N\to+\infty} \sup_{0\leqslant\leqslant L} |f(x) - S_N(x)| = 0,$$

holds for the partial sums $S_N(x)$ of Fourier sine series,

$$S_N(x) = \sum_{m=1}^{\infty} b_m \sin(m\pi x/L),$$

where

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx.$$

Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function satisfying $f'(0) = 0$. Then,

$$\lim_{N\to+\infty} \sup_{0\leqslant\leqslant L} |f(x) - S_N(x)| = 0,$$

holds for the partial sums $S_N(x)$ of Fourier cosine series,

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x/L),$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, \qquad a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx.$$

4. Review: ODE

We recall the some well-known results in ODEs. We will also use them without proofs.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u''(x) + \mu^2 u(x) = 0. \tag{11}$$

Then,

$$u(x) = c_1 \sin(\mu x) + c_2 \cos(\mu x), \tag{12}$$

for some constants c_1, c_2 depending on initial (or boundary data). For example, if $u(x)$ satisfies $u(0) = 0$ and $u'(0) = 1$, then the constants must be $c_1 = \mu^{-1}$ and $c_2 = 0$.

Suppose that a function $u(x)$ satisfies the following differential equation

$$u'(x) = \lambda u(x). \tag{13}$$

Then,

$$u(x) = ce^{\lambda x}, \tag{14}$$

for some constant c depending on the initial data.

5. Separation of Variables

In this section, we will SOLVE the Cauchy-Dirichlet problem with the vanishing Dirichlet data. Namely, given smooth $g(x)$, we will find the solutions to the heat equation (1) under the conditions (2) and (3), where $h_1(t) = h_2(t) = 0$.

To begin with, we remind that by the uniqueness theorem 1 there exists at most one solution. Hence, if we find a solution, then it is the only solution.

Next, we want find a function $u(x,t) = v(x)w(t)$ satisfying (1) and (3) with $h_1 = h_2 = 0$. (*Notice that in this step we do not consider* (2)*, yet.*) Then, (1) implies

$$w_t v = u_t = u_{xx} = w v_{xx}.$$

Dividing by vw yields

$$\frac{w_t(t)}{w(t)} = \frac{v_{xx}(x)}{v(x)}.$$

The left hand side only depends on t, while the right hand side only depends on x. Therefore, there exists some constant $\lambda \in \mathbb{R}$ such that

$$\frac{w_t}{w} = \frac{v_{xx}}{v} = \lambda.$$

We consider the three cases that $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$.

Case 1: $\lambda > 0$. In this case, by using the Dirichlet condition $v(0) = v(L) = 0$ we can obtain

$$0 \leqslant \lambda \int_0^L v^2 dx = \int_0^L v(\lambda v)dx = \int_0^L v v_{xx} dx = v v_x\big|_0^L - \int_0^L |v_x|^2 dx = -\int_0^L |v_x|^2 dx \leqslant 0, \tag{15}$$

namely $v = 0$. Thus, $u = 0$.

Case 2: $\lambda = 0$. In this case, $v_{xx} = 0$ implies $v(x) = ax + b$. Hence, the Dirichlet condition $v(0) = v(L) = 0$ guarantees $v = 0$. Thus, $u = 0$.

Case 3: $\lambda = -\mu^2 < 0$. In this case, the equation $v_{xx} + \mu^2 v = 0$ has non-trivial solutions. By the results in ODE, $v(x) = A\cos(\mu x) + B\sin(\mu x)$ holds for the constants A, B satisfying the boundary conditions

$$0 = v(0) = A\cos 0 + B\sin 0 = A,$$
$$0 = v(L) = A\cos(\mu L) + B\sin(\mu L) = B\sin(\mu L).$$

Hence, we have $\sin(\mu L)$, and thus $\mu L = m\pi$ for a natural number m. Namely, given $m \in \mathbb{N}$ we have

$$v_m = c\sin(m\pi x/L),$$

for some constant c. In addition, $\lambda = -\mu^2 = (m\pi/L)^2$ gives

$$\tfrac{d}{dt}w_m = -\mu^2 w_m = -(m\pi/L)^2 w_m. \tag{16}$$

Hence, the ODE result says

$$w_m = c\exp(-(m\pi/L)^2 t)$$

for some constant c. In conclusion, for each $m \in \mathbb{N}$ and any constant $B_m \in \mathbb{R}$

$$u_m(x,t) = B_m \exp(-(m\pi/L)^2 t)\sin(m\pi x/L)$$

satisfies (1) and (3) with $h_1 = h_2 = 0$.

By the result in the last case, we know that

$$u = \sum_{m=1}^{\infty} B_m \exp(-(m\pi/L)^2 t)\sin(m\pi x/L), \tag{17}$$

satisfies (1) and (3) with $h_1 = h_2 = 0$.

Now, we define the coefficients B_m by

$$B_m = \frac{2}{L}\int_0^L g(x)\sin(m\pi x/L)dx. \tag{18}$$

Then, by the Fourier series theorem above, the function $u(x,t)$ in (17) satisfies (2). Namely, it is the desired solution.